Ecole Doctorale Carnot-Pasteur

Proposition de sujet de thèse

Intitulé français du sujet de thèse proposé :

Fibrés instanton sur les variétés de dimension supérieure

Intitulé en anglais du sujet de these proposé :

Instanton bundles on higher dimensional manifolds

Unité de recherche :

IMB (UMR 5584, Université Bourgogne Europe & CNRS)

Nom, prénom et courriel du directeur (et co-directeur) de thèse :

Daniele Faenzi
daniele.faenzi@ube.fr

Domaine scientifique principal de la thèse :

Géométrie algébrique

Domaine scientifique secondaire de la thèse :

Algèbre commutative, algèbre homologique

Description du projet scientifique :

Moduli spaces of holomorphic and algebraic sheaves / vector bundles on projective manifolds have been studied from many perspectives, including character varieties and representations of the fundamental group, invariant theory, sheaf-theoretic methods, and higher categorical approaches.

When the underlying manifold has dimension 1, these moduli spaces enjoy very pleasant properties: they are integral, projective Fano varieties with mild singularities; their Betti numbers and deformation theory are well understood—even though their explicit geometric description is often elusive.

In dimension 2, these moduli spaces are also well behaved in an asymptotic region of the characteristic classes (results of O'Grady), and their geometry can often be viewed as a higher-dimensional analogue of the geometry of the base surface. For instance, for K3 surfaces the moduli space of stable sheaves is a holomorphic symplectic manifold whose cohomology lattice is controlled by the intersection lattice of the surface.

However, when the underlying complex / algebraic manifold has dimension 3 or more, the geometry of these moduli spaces becomes much more mysterious, and many basic questions remain unanswered. For instance, almost nothing is known about non-emptiness, connectedness, irreducibility, singularities, or (stable) rationality of these moduli spaces, let alone about finer invariants such as Hodge numbers, the automorphism or birational group, the Kodaira dimension, or the structure of the Mori cone or of the derived category of coherent sheaves.

Instanton bundles play a distinguished role in this framework. They form a special class of sheaves inspired by instantons in mathematical physics, seen as connections representing solutions of the Yang–Mills equations.

The structure of the moduli space of instantons has been exploited in the celebrated work of Donaldson on real 4-manifolds, while the algebro-geometric counterpart has deep connections with hyperkähler geometry, representation theory of quivers, and derived categories, as shown in the influential work of Atiyah, Drinfeld, Hitchin, Kronheimer, Manin, and Nakajima. From a different perspective, moduli of instanton sheaves also play a significant role in the structure of the base variety. For instance, for Fano threefolds the non-trivial piece (the Kuznetsov component) of the derived category is often described in terms of instanton bundles; moreover, moduli spaces of the variety itself are sometimes related to moduli of instantons, see [KMM10].

The thesis project aims at studying instanton sheaves and their moduli spaces, essentially in dimension 3 or higher. For threefolds, our understanding of instanton bundles is essentially limited to Fano threefolds of Picard rank one, where such bundles are known to exist provided that their topological invariants lie in a certain natural range, see [CF25]. Some features of their moduli spaces have also been established, relying on the structure of the Kuznetsov component of the threefold.

In this setting, an overarching goal of the thesis would be to understand instantons and their moduli spaces in more general dimensional situations, as well as their connections with other classes of vector bundles, such as Ulrich bundles introduced in the foundational paper [ES03].

A natural starting benchmark would be that of Fano threefolds of higher Picard number, where tools from birational geometry should allow effective constructions of curves and vector bundles inspired by ’t Hooft instantons, recently studied on the Del Pezzo threefold flag *F(1,2,3)*, see [AMMP-L].

The main focus of the thesis would be on fourfolds, notably hypersurfaces and Fano fourfolds. For cubic fourfolds, the relationship between instanton bundles, their acyclic extensions, and Ulrich bundles has been studied in detail through deformations of sheaves and objects of the Kuznetsov category, which in this case is a K3 category, see [CFG25]. This is a remarkable feature of cubic fourfolds, shared (sometimes conjecturally, e.g. in the case of Küchle fourfolds) with other manifolds such as Gushel–Mukai varieties. Some tools from representation theory are expected to apply to the construction of instanton and Ulrich bundles on these varieties and potentially on many—more ambitiously, all—Mukai varieties (Fano varieties of dimension n whose Fano index is $n-3$). As an intermediate step, it would be natural to construct and parametrize bundles with prescribed numerical invariants on projective bundles and blow-ups.

To extend and strengthen this approach, it would be interesting to develop a full treatment of the G-equivariant setup for the problems mentioned above, where G is an algebraic group acting on the base variety. More generally, the theory of G-actions on moduli spaces of vector bundles would benefit from solid foundational work. For instance, *SL(2)*-equivariant instantons play an important role in the construction of Ulrich bundles on Veronese threefolds. Understanding these equivariant bundles could have broader applications: for example, the equivariant instanton bundles mentioned above have recently been used in the study of moduli spaces of binary forms, as well as in the classification of equivariant spaces of matrices of constant rank. This could also have an impact on the study of algebraic subgroups of birational automorphism groups of varieties, such as the Cremona group, according to work in progress of D. Faenzi and R. Terpereau.

It should be noted that some of the methods developed for the geometric study of moduli spaces—notably the so-called orbital degeneracy loci—are also very useful for understanding moduli spaces of bundles on curves and surfaces. This is particularly visible in the context of Vinberg’s theta groups, in view of their connection with abelian varieties of small dimension. Indeed, some of these spaces appear as degeneracy loci associated with actions with finitely many orbits arising from gradings of Lie algebras. This has been extensively explored in [BMT21] and [BBFM26], as well as in [GS15], notably in connection with Coble hypersurfaces, which are special Heisenberg-invariant polynomials typically singular along polarized abelian varieties.

More recent work of Laga–Romano, based on earlier results of Thorne, opens the way to a more general treatment of Coble hypersurfaces and their moduli spaces, also known as Göpel varieties. This is another direction that could naturally be explored within the thesis.

Bibliographie

[AMMP-L] V. Antonelli, F. Malaspina, S. Marchesi and J. Pons-Llopis, 't Hooft bundles on the complete flag threefold and moduli spaces of instantons. J. Math. Pures Appl. (9) 202, Article ID 103763, 44 p. (2025).

[BBFM26] V. Benedetti, M. Bolognesi, D. Faenzi, and L. Manivel, *Hecke cycles on moduli of vector bundles and orbital degeneracy loci.* J. Algebr. Geom. 35, No. 1, 163-195 (2026).

[BMT21] V. Benedetti, L. Manivel and F. Tanturri, *The geometry of the Coble cubic and orbital degeneracy loci*. Math. Ann. 379, No. 1-2, 415-440 (2021).

[CFG25] G. Casnati, D. Faenzi and F. Galluzzi, *Ulrich and instanton bundles on special cubic fourfolds*, arXiv eprint math.AG/2511.14470.

[CF25] G. Comaschi and D. Faenzi, *Higher rank instantons sheaves on Fano threefolds*, arXiv eprint math.AG/2504.13505, 2025.

[ES03] D. Eisenbud and F.-O. Schreyer, *Resultants and Chow forms via exterior syzygies.* Appendix by J. Weyman. J. Am. Math. Soc. 16, No. 3, 537-575, appendix 576-579 (2003).

[GS15] L. Gruson and S. V. Sam, *Alternating trilinear forms on a ninedimensional space and degenerations of (3, 3)-polarized Abelian surfaces,* Proc. Lond. Math. Soc. (3) 110 (2015), no. 3, 755–785.

[KMM10] A. Kuznetsov, L. Manivel, and D. Markushevich, *Abel-Jacobi maps for hypersurfaces and noncommutative Calabi-Yau's*, Commun. Contemp. Math. 12 (2010), no. 3, 373–416.

Connaissances et compétences requises :

Des compétences solides sont requises concernant les variétés algébriques affines et projectives, les schémas, les faisceaux cohérents et leur cohomologie, aussi bien que sur la cohomologie des variétés complexes et de quelques constructions élémentaires de telles variétés (grassmanniennes etc). Des connaissances sur les classes caractéristiques des faisceaux, sur la géométrie birationnelle, les catégories dérivées, les espaces de modules, seront les bienvenues.