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# Nonlinear dispersive PDEs: Integrable structures and critical phenomena

Mémoire pour l'obtention de habilitation à diriger des recherches présentée par

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# Chapter 1

# Introduction

This work reviews the main part of my scientific efforts in the last ten years and, as a whole presents different takes on one overarching theme: dispersive partial differential equations. Motivation to study these equations, at least for me, always comes from physics: they model nonlinear phenomena in applications from water waves to hot plasma and from Bose-Einstein condensates to optical phenomena. One small subset of these equations is particularly fascinating: integrable systems. There is no generally accepted definition of what an integrable system is, in most cases the features of these systems are called upon to describe them. These include, but are not limited to, existence of infinite families of exact solutions, the existence of infinitely many commuting flows and conserved quantities, or a transformation of the system, under certain conditions, to a linear system. In the case of dispersive systems there is a balance of the linear terms, that try to disperse a localised mass, and the nonlinear terms, that try to concentrate it, so that these two forces are in an equilibrium, and allow for infinite families of exact solutions.

The exposition in the present work follows the timeline in which I worked on different topics quite closely. For me this is not only convenient, but also a natural way to present the topic: as my understanding grew I have looked into questions that arose along the way. The exposition is arranged along different ideas and methods that are used in the study, however these remain different perspectives on the same subject. We begin with hydrodynamic type systems, also called dispersionless systems. This area has somewhat simple origins and can be looked at as a development of possibly the simplest PDEs: a first order equation with constant coefficients in two dimensions

$$u_t - au_x = 0.$$

Throughout this work I use the standard short notation, where partial derivatives are denoted by subindicies and the independent variables are generally omitted unless necessary i.e. $u = u(x,t)$. The above equation can be solved by the method of characteristics to find that $u = f(x - at)$ for every function $f$. Things become somewhat more interesting when

we consider a *nonlinear* equation, for example the Hopf equation

$$u_t - uu_x = 0. \tag{1.1}$$

Here the general solution, is given by $u = f(x - ut)$ and it is an implicit solution, which cannot be made explicit for $f$ in general position. As a generalisation, let us consider a system of nonlinear, first order equations, for example describing an ideal barothropic gas,

$$\rho_t + (\rho u)_x = 0, \qquad u_t + uu_x + p(\rho)\rho_x = 0,$$

where $p = p(\rho)/\rho$ is the equation of state. This system can be written in differential form as

$$\mathrm{d}\rho \wedge \mathrm{d}x + \rho \mathrm{d}t \wedge \mathrm{d}u + u\mathrm{d}t \wedge \mathrm{d}\rho = 0 \qquad \mathrm{d}u \wedge \mathrm{d}x + u\mathrm{d}t \wedge \mathrm{d}u + p(\rho)\mathrm{d}t \wedge \mathrm{d}\rho = 0.$$

If we now exchange the dependent and independent variables, namely take $t = t(u, \rho)$ and $x = x(u, \rho)$, and differentiate we arrive at

$$(-x_u - \rho t_\rho + ut_u)\ \mathrm{d}u \wedge \mathrm{d}\rho = 0 \qquad (x_\rho - ut_\rho + p(\rho)t_u)\ \mathrm{d}u \wedge \mathrm{d}\rho = 0, \tag{1.2}$$

namely a linear system with variable coefficients, for which we can find a general solution. The exchange of dependent and independent variables is often called a *point* transformation, a change that also involves first derivatives is called a *contact* transformation; for example the Burgers equation $u_t + uu_x - u_{xx} = 0$ is linearisable by making the change $v = u_x/u$. Systems that can be linearised by a point or a contact change of variables are called *C-integrable* in contrast to *S-integrable* systems which can be transformed to action-angle variables through an *inverse scattering transform*.

There are two main questions addressed in this work. First, can we classify all hydrodynamic type systems that are integrable, at least for some large families. Second, having an dispersionless system that is integrable, can we reconstruct (possibly all) dispersive systems which have it as a limit. For example, if we take the famous Korteweg-de Vries equation,

$$u_t - uu_x - \epsilon^2 u_{xxx} = 0,$$

where $\epsilon$ can be seen as a small parameter controlling dispersion, it is quite easy to take the limit $\epsilon \to 0$ and obtain the Hopf equation (1.1). So the question is, can this be inverted and can we recover integrable dispersive equations from their hydrodynamic type (dispersionless) limits.

In (1+1) dimensions, that is, for systems

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0, \tag{1.3}$$

where $\mathbf{u}$ is an $m$-dimensional vector and $\mathbf{A}$ an $m \times m$ matrix The classification question was answered by Sergey Tsarev [64], by introducing the semi-Hamiltonian property as

a criterion for integrability. The question about dispersive deformations was studied at length by Boris Dubrovin [15, 16, 17, 18], together with the behaviour of solutions, leading to the *universality* conjecture as well as far reaching results such as Frobenius manifolds.

In (2+1) dimensions, namely systems of the type

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x + \mathbf{B}(\mathbf{u})\mathbf{u}_y = 0, \tag{1.4}$$

and to some extend higher dimensions Jenya Ferapontov introduced the *hydrodynamic reductions* and their dispersive deformations [23].

My contribution to the field starts with the classification of all two-component integrable Hamiltonian systems of hydrodynamic type in $2+1$ dimensions (1.4) [30] and their dispersive deformations [29].

For two component systems in $2+1$ dimensions, Poisson brackets fall into three classes. For each of these classes we obtain a full list of integrable Hamiltonian densities.

The central result of this research is that the moduli spaces of integrable Hamiltonians in $(2+1)$ dimensions are finite dimensional, with one 'generic' Hamiltonian for each class of Poisson brackets, as well as a number of 'degenerate' potentials, including some polynomial densities. For some of the more complicated cases the solution was found only in parametric form, with parameters being solutions of generalised hyper-geometric equations [30].

We show that for Hamiltonian systems of hydrodynamic type non-trivial dispersive deformations exist only for certain cases, but not in general. This is in contrast with the situation in $(1+1)$ dimensions where the moduli space of integrable Hamiltonians is parametrised by a number of functions and every such system is deformable [29].

In [55] we demonstrate that for high-dimensional systems, for example $(3+1)$ or $(4+1)$ systems, the method of hydrodynamic reductions can be augmented to include higher than $(1+1)$ dimensional reductions, effectively extending the class of systems that can be studied. Our study is motivated by examining the hydrodynamic limit of Bogoyavlenskii's equations.

In [56] we present a schema to understand the WDVV associativity equations in the case $N = 3$ as high-frequency limit of a three component "intermediate" dispersive system. This "intermediate" system is of Camassa-Holm type and shares its dispersionless limit with the Yadjima-Oikawa system, and thus they are in the same universality class.

Chapter Two is devoted to a more detailed review of the topic, as well as a systematic description of my results. The chapter also includes a section of bidifferential calculus.

The relationship between dispersive equations and their hydrodynamic type limit cannot be fully understood without looking at solutions and what happens to them at, and close to this limit. The leading concept in this is Dubrovin's *universality* conjecture, which, briefly, claims that all solutions of Hamiltonian equations that have the same dispersionless limit behave in exactly the same way up to and around the critical point of the solution

of the dispersionless equation. The conjecture has been proven for a number of scenarios in (1+1) dimensions. This regime is known as small dispersion limit, where one can introduce a small parameter $\epsilon$. What is implicit in the definition is that making the dispersion small we let the non-linearity be much larger, thus we are actually looking at the strong non-linear coupling regime where we can find different kinds of critical phenomena. This includes *dispersive shock waves*, which are zones of rapid oscillations near the critical point, soliton resolution, where some general initial data are resolved into solitons and radiation and the formation of singularities from smooth initial data. As solutions cannot be found explicitly, numerical simulation can play a major role in understanding critical phenomena and producing conjectures that can guide analytical work. This is especially true in higher dimensions where analytical results are scarce.

In the second part of the thesis I describe numerical work, using spectral collocation methods to address various problems in mathematical physics, in the theory of PDEs and special functions, in mathematical applications to physics and medical imaging and so on. Although these fields may seem somewhat scattered they are connected with much more than the common computational methods we use. The central idea is that spectral collocation methods, that is approximating functions via sets of global functions, have overwhelmingly more advantages than disadvantages compared to other methods. Basic examples for spectral discretisation are approximating periodic functions via truncated Fourier series and approximating finitely supported functions by Chebyshev polynomials. The principal advantage is the so called spectral : for functions that are analytic on the computational domain, the numerical error decreases *exponentially* with the number of collocation points. Double machine precision of 15 orders of magnitude, which most of today's computers use as a standard can be achieved with between tens to thousands of points.

Spectral method are not universally popular for three main reasons: First, they are excellent at approximating regular functions, ideally analytic. The performance sharply degrades once the functions are less regular. In a typical example a spectral Fourier based method is only of order $n$ for $C^n$ functions. The second disadvantage which derives from the first, is that spectral methods well work only on relatively simple computational domains, where boundaries are coordinate surfaces. Third, they are computationally intensive. As we shall see, differentiation matrices are global in the computational domain and require dense matrix operations. One further consequence of this is that an error in one grid point spreads on the entire computational domain.

We can compare this to the standard finite difference/ finite element/ finite volume methods (FDM, FEM, FVM respectively). There the computational domain is divided into a number of sub-domains (elements) where derivatives are computed via next-neighbour difference (FDM), approximation with low order polynomials (FEM), or locally defined smooth functions (FVM). This provides great flexibility, as complicated shapes can be discretised via triangulation without too much trouble. The drawback is the relatively low precision: spatial discretisation is usually up to second order which means machine precision

may be difficult to obtain as it may require a billion points in each spatial direction. Such precision, however, is never needed in engineering applications and as a result the FEM and FVM are the bread and butter of all engineering modelling and computation, from structural design to computational fluid dynamics. The same to a certain extent is true for spatial discretisation schemes in computational physics, medical imaging and so on.

In mathematical and physical setting the unmatched precision of spectral methods can be exploited with careful analytical consideration of the problem, the use of multi-domain approaches and compactification of external domain. Emerging high-performance mass-parallel computing techniques allow to scale and accelerate the codes. This combination provides insights and guidance in areas that are unattainable with analytical methods or less sophisticated computing. Examples include critical phenomena in multi-dimensional PDEs, special functions, inverse problems and so on.

In a series of articles we numerically explore the Davey-Stewartson (DS) equation. In [42] we treat DS as an evolutionary PDE and study its blow-up mechanism for smooth initial data, conjecturing the type of blow-up and rates of different norms. In [33] we develop a high-precision *hybrid* method for DS using explicit analytic regularisation combined with a Fourier-based spatial resolution.

In [34, 43] and [40, 41] we look at DS from the standpoint of integrable systems, addressing the direct and inverse scattering problems. They are both described by a system of d-bar equations, which is the central object of study. In [34] we look at Schwartz class potentials, analytically regularising the singular terms and then applying a Fourier discretisation. In [43] we study compactly supported functions via a multi-domain Chebyshev-based discretisation. This is further developed in [40] where we present results on the asymptotic behaviour for large values of the spectral parameter $k$ of the d-bar system and complement it with a computational approach for low and intermediate values of the same parameter, thus developing a comprehensive approach.

In [36] and [37] we look at critical phenomena in the Zakharov-Kuznetsov (ZK) family of equations. This equation is not integrable, however it has some familiar properties: it has several conserved quantities, has soliton solutions, and so on. We study soliton stability and resolution, development of singularities from smooth initial date and the formation of dispersive shock waves.

In the articles [8] and [44] and [46] we propose methods for computing the hypergerometric function and some Painlevé transcendents and the Hilbert transformations to machine precision on selected subsets of the complex plane using global spectral methods, achieving a significant improvement in terms of speed, efficiency and accuracy over existing approaches. We would like to make two remarks here. First in the theory of special functions, a function is usually regarded as known when it is connected to a certain ODE and its automorphic properties and relations to other special functions are known. This, however does not mean that its values are actually known: for this we need a computational method. For most functions standard software packages can reliably provide values

to about eight orders of magnitude, which is sufficient for most applications, but still away from machine precision. This leads us to the second remark: we intend to use these works and results in further research in PDEs, where controlling the precision when studying critical phenomena is crucial.

A direct application of these numerical techniques in physics is an ongoing joint project between the Physics and Mathematics Institutes at the University of Burgundy where we study the optical properties of a spheroidal quantum dot coupled to a metallic nanoparticle. As the standard photon Green's function expansions does not work well for this geometry, we implement a direct PDE solver based on spectral methods over finite domains.

Some of my other works, that will not be discussed in detail in the present text include bidifferential calculus, economic modelling for engineering applications, and exact solutions in the AdS/CFT correspondence.

# Chapter 2

# Systems of hydrodynamic type

In this chapter we present some notions of the theory of hydrodynamic type equations and their integrability. We begin with $(1+1)$ dimensional systems and describe notions of Hamiltonian structure for infinite dimensional systems. WE continue with higher dimensional theory, which has only been develop recently and where examples are fewer. Finally, we present the construction of dispersive deformations, namely how to reconstruct dispersive integrable systems from their dispersionless limits.

## 2.1 Hydrodynamic type systems

We start with basic notions on systems of hydrodynamic type, namely

$$u^i_t = \sum_{j=1}^{n} v^i_j(\mathbf{u}) u^j_x, \tag{2.1}$$

are called quasi-linear systems in $1+1$ dimensions, or systems of hydrodynamic type. Here $\mathbf{u} = (u^1(t,x), u^2(t,x), \ldots, u^n(t,x))$ is a $n$-component vector of dependent variables. The functions $v^i_j(\mathbf{u})$, which could also be considered as matrix elements of an $n \times n$ matrix $\mathbf{V}$, are assumed sufficiently smooth and, in general, non-constant. Systems of this type arise in many backgrounds including differential geometry, general relativity, magneto-fluid dynamics, etc. The system (2.1) is called *strictly hyperbolic* iff all eigenvalues of $\mathbf{V}$ are real and distinct.

If there exists a local change of variables $u^i = u^i(R^1, R^2, \ldots, R^n)$, such that the matrix $\mathbf{V}$ is diagonal in the new coordinates $R^i$ then system (2.1) is diagonalisable. The $R^i$ coordinates are called Riemann invariants, and in this new set of variables the system has the form

$$R^i_t = v_i(\mathbf{R}) R^i_x. \tag{2.2}$$

There exists an invariant differential-geometric criterion for diagonalisability due to Haantjes [32]. One constructs the Nijenhuis tensor from a strictly hyperbolic matrix $\mathbf{V} =$

$v^i_j$,

$$N^i_{jk} := \sum_{p=1}^{n}\sum_{q=1}^{n} v^s_j \partial_s v^i_k - v^s_k \partial_s v^i_j - v^i_s(\partial_j v^s_k - \partial_k v^s_j),$$

and then uses it to find the Haantjes tensor,

$$H^i_{jk} := \sum_{p=1}^{n}\sum_{q=1}^{n} (N^i_{qp} v^q_k - N^q_{kp} v^i_q) v^p_j - v^i_p (N^p_{qj} v^q_k - N^q_{kp} v^i_q).$$

The following theorem then holds

**Theorem 1.** *[32] A strictly hyperbolic matrix $v^i_j$ is diagonalisable if and only if the corresponding Haantjes tensor $H^i_{jk}$ is identically zero.*

Hamiltonian structure can be defined via a Poisson bracket of the following form:

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} A^{ij} \frac{\delta J}{\delta u^j(x)} dx. \tag{2.3}$$

Here $A^{ij}$ is a Hamiltonian differential operator

$$A^{ij} = g^{ij}(\mathbf{u}(x,t))\frac{d}{dx} + b^{ij}_k(\mathbf{u}(x,t))u^k_x.$$

The following important theorem, due to Dubrovin and Novikov then holds:

**Theorem 2.** *[12]*

1. *Under local changes of coordinates, $g^{ij}$ transforms like a (2,0) - tensor. Furthermore, if $\det g^{ij} \neq 0$ then $b^{ij}_k$ transforms like the expression $-g^{is}\Gamma^i_{sk}$, where $\Gamma^i_{sk}$ is an affine connection.*

2. *The skew-symmetry of (2.3) imposes that $g^{ij}$ must be symmetric and thus it can be considered as a pseudo-Riemanian metric on the space of field variables $\mathbf{u}$. Moreover, the connection $\Gamma^i_{sk}$ must be compatible with this metric, $\nabla_k g_{ij} = 0$.*

3. *In order that the bracket (2.3) satisfies the Jacoby identity,*

$$\{\{I, J\}, K\} + \{\{J, K\}, I\} + \{\{K, I\}, J\} = 0,$$

*it is necessary and sufficient that the connection $\Gamma^i_{sk}$ is torsionless and flat i.e. $\Gamma^i_{sk} = \Gamma^i_{ks}$ and the Riemann curvature tensor $R^i_{jkl} = 0$.*

A Hamiltonian of hydrodynamic type is a functional $H = \int h(\mathbf{u})dx$ such that its density $h(\mathbf{u})$ depends only on the field variables $\mathbf{u}(x,t)$, but is independent of their spatial derivatives $\mathbf{u}_x$, $\mathbf{u}_{xx}$, .... A Hamiltonian system of hydrodynamic type $u^i_t = \{u^i, H\}$ then explicitly is

$$u^i_t = \left( g^{ik} \frac{\partial^2 h}{\partial u^k \partial u^j} + b^{ik}_j \frac{\partial h}{\partial u^k} \right) u^j_x.$$

Strictly hyperbolic, diagonal Hamiltonian systems are integrable by the following geometric construction: Let

$$u^i_t = v_i(\mathbf{u}) u^i_x.$$

Then,

$$0 = \nabla_i v^k_i - \nabla_j v^k_i = \partial_i v_j \delta^j_k - \partial_j v_i \delta^i_k + \Gamma^k_{ij}(v_j - v_i).$$

This means that $\Gamma^k_{ij} = 0$ for $i \neq j \neq k$ and

$$\partial_i v_k = \Gamma^k_{ki}(v_i - v_k), \qquad i \neq k. \tag{2.4}$$

On the other hand, for diagonal matrices $v^i_j = v_i \delta^j_i$ the condition $\sum_k g_{ik} v^k_j = \sum_k g_{jk} v^k_i$ directly implies that the associated metric $g_{ik}$ is diagonal since,

$$\sum_{k=1}^{n} g_{ik} v^k_j - g_{jk} v^k_i = g_{ij}(v_j - v_i) = 0,$$

and $v_i \neq v_j$ for $i \neq j$ due to hyperbolicity. From $\Gamma^k_{ij} = \sum_s \frac{1}{2} g^{ks}(\partial_j g_{si} + \partial_i g_{sj} - \partial_s g_{ij})$ we find

$$\Gamma^k_{ki} = \frac{1}{2} \partial_i \ln g_{kk}. \tag{2.5}$$

From the last equation it directly follows that the consistency condition $\partial_j \partial_i \ln g_{kk} - \partial_i \partial_j \ln g_{kk} = \partial_j \Gamma^k_{ki} - \partial_i \Gamma^k_{kj} = 0$ is equivalent to

$$\partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right), \qquad i \neq j \neq k. \tag{2.6}$$

The solutions of systems of type (2.4) are discussed in classical differential geometry [9]: The existence of a flat diagonal metric is equivalent to the existence of an orthogonal curvilinear coordinate system in (pseudo-) Euclidean space. If we are given an orthogonal curvilinear system of coordinates it is natural to ask how to find the relevant Hamiltonian matrices $v^i$. Evidently, from the arguments in this section, the sole requirement for a diagonal matrix to be Hamiltonian and respectively possess an associated flat metric is (2.4).

On the other hand, we can think of (2.4) as a linear system of equations for $v_i(\mathbf{u})$, which is compatible if and only if

$$\partial_i \Gamma^k_{kj} - \Gamma^k_{kj} \Gamma^j_{jj} - \Gamma^k_{ki} \Gamma^i_{ij} + \Gamma^k_{ki} \Gamma^k_{kj} = R^k_{jik} = 0, \tag{2.7}$$

from which follows

$$\partial_j \Gamma^k_{ki} - \partial_i \Gamma^k_{kj} = R^k_{kij} = 0,$$

where $R^k_{kji}$ are components of the Riemann curvature tensor. These conditions are trivially satisfied for a flat metric $g_{ij}$.

System (2.4) is an $n$-component first order system, thus the general solution depends on $n$ functions of a single variable. This implies that for each orthogonal curvilinear coordinate system there exists a family of Hamiltonian matrices, locally parametrised by $n$ functions of a single variable.

The following theorem [64] summarises the results discussed in this section.

**Theorem 3.** *The metric $g$, associated with the Hamiltonian matrix $\mathbf{V}$, $v^i_j(\mathbf{u}) = v_j \delta^i_j$, is diagonal and the variables $\mathbf{u}$ constitute a curvilinear orthogonal system of coordinates. On the other hand for each curvilinear orthogonal system of coordinates there exists a family of Hamiltonian matrices, which are diagonal in this system of coordinates. This family is locally parametrized by $n$ functions of a single variable. For all matrices $\mathbf{V}$ with different diagonal coefficients $v_i$ belonging to this family the following relations hold:*

$$\partial_i v_k = \Gamma^k_{ki}(v_i - v_k), \qquad \partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right).$$

The Hamiltonian property of a hydrodynamic type system is sufficient for integrability, but can be weakened. The conditions (2.7) follow directly from (2.6), thus, given a hyperbolic diagonal system $u^i_t = v_i(\mathbf{u}) u^i_x$ with distinct characteristic speeds $v_i$, that satisfy (2.6) then the system of equations for $w_i$,

$$\partial_i w_k = \Gamma^k_{ki}(w_i - w_k), \quad \Gamma^k_{ki} = \frac{\partial_k v_i}{v_k - v_j},$$

is compatible. A diagonal hydrodynamic system $u^i_t = v_i u^i_x$ is called semi-Hamiltonian if it is hyperbolic and its coefficients satisfy the condition

$$\partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right).$$

The semi-Hamiltonian condition is necessary and sufficient for integrability.

**Theorem 4.** *[64] A semi-Hamiltonian system has infinitely many commuting flows, parametrised locally by $n$ functions of a single variable. These flows commute with each other, their matrices are all diagonal and all hydrodynamic type integrals of the initial semi-Hamiltonian system are also their integrals.*

The general solution of an $n$-dimensional semi-Hamiltonian diagonal system $u^i_t = v_i u^i_x$ can be solved by virtue of the generalised hodograph method [64], which is a multi-component generalisation of the method of characteristics. As we have already discussed,

a system of the considered type has infinitely many commuting flows $u^i_\tau = w_i u^i_x$, which satisfy the equations

$$\frac{\partial_i w_k}{w_i - w_k} = \frac{\partial_i v_k}{v_i - v_k}, \quad i \neq k.$$

Let us now construct a system of $n$ equations for the field variables $\mathbf{u}$,

$$w_i(\mathbf{u}) = v_i(\mathbf{u})t + x. \tag{2.8}$$

Here $x$ and $t$ are parameters, $v_i$ are the coefficients of the original semi-Hamiltonian matrix, and $w_i$ are the coefficients of any flow commuting with the original system. The following theorem then holds:

**Theorem 5.** *[64] Any smooth solution of (2.8) is a solution of the semi-Hamiltonian system $u^i_t = v_i u^i_x$. Furthermore, any solution of the given system $u^i_t = v_i u^i_x$ may be represented as a solution of (2.8) in a neighbourhood of a point $(t_0, x_0)$ such that $u^i_x(t_0, x_0) \neq 0$.*

## 2.2 Multi-dimensional extensions

The method of hydrodynamic reductions [23] provides a way to investigate the integrability of higher dimensional systems of hydrodynamic type. Consider a $2+1$ dimensional system of hydrodynamic type in the following form:

$$\mathbf{u}_t = \mathbf{A}(\mathbf{u})\mathbf{u}_x + \mathbf{B}(\mathbf{u})\mathbf{u}_y. \tag{2.9}$$

Here $\mathbf{u}$ is an $m$-component vector, $\mathbf{A}(\mathbf{u})$ and $\mathbf{B}(\mathbf{u})$ are $n \times n$ matrices, with $l$ being the number of equations. We consider the following ansatz

$$\mathbf{u}(x, y, t) = \mathbf{u}(R^1, R^2, \ldots, R^N),$$

with $R^i(x, y, t)$ required to satisfy the equations,

$$R^i_t = \lambda^i(\mathbf{R})R^i_x, \qquad R^i_x = \mu^i(\mathbf{R})R^i_x. \tag{2.10}$$

We can think of this as a decoupling of the original $2+1$ dimensional system (2.9) into two separate commuting $1+1$ dimensional systems of hydrodynamic type from where the name hydrodynamic reductions comes from. Substituting the ansatz into (2.9) we arrive at the following equation

$$(\lambda^i I - A - \mu^i B)\partial_i \mathbf{u} = 0, \tag{2.11}$$

which implies that $\lambda^i$ and $\mu^i$ must satisfy the dispersion relation $\det(\lambda I - A - \mu B) = 0$

We can now compute the commutativity condition $R_{ty} = R_{yt}$ to find that

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i} \quad i \neq j. \tag{2.12}$$

Provided (2.12) holds, the solution of the system (2.9) is given by the generalised hodograph formula

$$v^i(\mathbf{R}) = x + \lambda^i(\mathbf{R})t + \mu^i(\mathbf{R})y, \tag{2.13}$$

where $v^i$ is the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j. \tag{2.14}$$

The system (2.9) is integrable if it possesses sufficiently many $N$-component reductions, parametrized by $N$ functions of a single variable.

We note here that this definition agrees with any other definition of integrability for this kind of systems e.g. it posseses infinitely many higher commuting flows, it can be expressed as a compatilbility condition of a "dispersionless" Lax Pairs and so on [23, 26, 28].

## 2.3 Dispersive systems as integrable deformations

The deformation theory of $1+1$ dimensional Hamiltonian systems is based on the theory of integrability of hydrodynamic type Hamiltonian systems in $1+1$ dimensions, presented in Chapter 2. We recall that a Hamiltonian system of hydrodynamic type

$$u^i_t = \{u^i, H_0\} = P^{ij}\ \delta H_0/\delta u^j, \tag{2.15}$$

$i, j = 1, \ldots, n$, where $P^{ij} = \epsilon^i \delta^{ij} d/dx$ is the Hamiltonian operator and $H_0 = \int h(u)\ dx$ is the Hamiltonian with the density $h(u)$, is integrable if and only if it is diagonalisable. Integrability in this case is understood as the existence of infinitely many functionals $F = \int f(u)\ dx$, commuting with the Hamiltonian, $\{H, F\} = 0$. The functionals $F$ are parametrized by $n$ arbitrary functions of one variable.

There are several approaches in studying deformations of integrable dispersionless equations. These include deformations of Lie algebra homomorphisms [21] and dressing operator method applied to Moyal algebra valued loop group [62]. We will follow the route taken by Boris Dubrovin and his collaborators in a series of papers [16, 17, 18, 19, 20]. Given an integrable Hamiltonian system of hydrodynamic type one considers a deformation of the original Hamiltonian in the form

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \tag{2.16}$$

where the density of $H_i$ is assumed to be a homogeneous differential polynomial of degree $i$ in the $x$-derivatives of $\mathbf{u}$ with coefficients being functions of $u^i$ themselves. Here the Hamiltonian operator $P^{ij}$ can be assumed undeformed due to the general results of [31]. Deformation (2.16) is called integrable (to the order $\epsilon^m$) if any hydrodynamic Hamiltonian $F_0 = \int f(u)\ dx$ commuting with $H_0$ can be deformed in such a way that

$\{H, F\} = 0 \pmod{\epsilon^{m+1}}$. The classification of integrable deformations is performed modulo canonical transformations of the form

$$H \to H + \epsilon\{K, H\} + \frac{\epsilon^2}{2}\{K, \{K, H\}\} + \dots \tag{2.17}$$

where $K$ is any functional of the form (2.16). The richness of this deformation scheme is due to the following facts: First, the variety of integrable 'seed' Hamiltonians $H_0$ is parametrised by $n(n-1)/2$ arbitrary functions of two variables. Second, for a fixed integrable Hamiltonian $H_0$, the deformation procedure introduces extra arbitrary functions of one variable known, in bi-Hamiltonian context, as 'central invariants'. One should point out that it is still an open problem to extend a deformation, for arbitrary values of these functions, to all orders in the deformation parameter $\epsilon$.

An analogous deformation scheme in $2+1$ dimensions, was developed in the context of hydrodynamic reductions. We will say that a deformed system is integrable (integrable to order $\epsilon^m$) iff all its hydrodynamic reductions inherited from the original system could be deformed without a breach of the commutativity conditions. We note here that the Hamiltonian operator will be left undeformed. We will later present an example in (1+1) dimensions where the deformation can be moved between the Hamiltonian operator and the potential, however we are not aware of any results establishing the triviality of Poisson cohomology in higher dimensions.

## 2.4 Results

### 2.4.1 Classification of integrable hamiltonian systems and their dispersive deformations

In the 1+1 dimensional case, for fixed signature there exists only one non-degenerate Poisson bracket. In contrast, in the two component 2+1 dimensional case there are three essentially different types of Poisson brackets, namely

$$P_I = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix}, \quad P_{II} = \begin{pmatrix} 0 & d/dx \\ d/dx & d/dy \end{pmatrix},$$

whereas the third one is essentially non constant:

$$P_{III} = \begin{pmatrix} 2v & w \\ w & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & v \\ v & 2w \end{pmatrix} \frac{d}{dy} + \begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix}.$$

We show that the moduli spaces of integrable Hamiltonians in $(2+1)$ dimensions are finite dimensional, with one 'generic' Hamiltonian for each class of Poisson brackets, as well as a number of 'degenerate' potentials, including some polynomial densities. For some of the more complicated cases the solution was found only in parametric form, with parameters

being solutions of generalised hyper-geometric equations [30]. Further, when considering deformations, we show that for Hamiltonian systems of hydrodynamic type non-trivial dispersive deformations exist only for certain cases, but not in general. This is in contrast to the situation in $(1+1)$ dimensions where the moduli space of integrable Hamiltonians is parametrised by a number of functions and every such system is deformable [29].

### 2.4.2 Exceptional cases: WDVV

There exist several examples of hydrodynamic type systems that are not diagonalisable, but are nevertheless integrable [27, 22]. Most of them are connected to the three-wave equation and are integrable by inverse scattering transform, what is sometimes called $S$-integrable. In contrast standard (i.e. diagonalisable) hydrodynamic type systems are $C$-integrable by some complicated contact change of coordinates and use of implicit functions. Among these systems is one of the Witten-Djikgraaf-Verlinde-Verlinde (WDVV) equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}.$$

Introducing $w^1 = f_{xxx}$, $w^2 = f_{xxt}$ and $w^3 = f_{xtt}$ we can write it in quasilinear form

$$w1_t = w^2_x, \quad w^2_t = w^3_x, \quad w^3_t = [(w^2)^2 - w^1 w^3]_x. \tag{2.18}$$

This equation appears in Topological Field Theory; in the context of classical differential geometry this equation describes the Egorov orthogonal curvilinear coordinate nets, see [15] and references therein. This linearly degenerate hydrodynamic type system is non-diagonalisable, thus it is not semi-Hamiltonian, but is nevertheless integrable by the inverse scattering transform. This means that system (2.18) is not a dispersionless limit of some integrable dispersive system. In [55] we construct the dispersive system

$$\begin{gathered}(1+\epsilon\partial_x)a_t = (a^2 + 2b + \epsilon a a_x + \epsilon b_x)_x, \\ b_t = (-2c - \epsilon b a_x + \epsilon c_x)_x, \\ (1-\epsilon\partial_x)c_t = (2ac - \epsilon b b_x - \epsilon a c_x)x.\end{gathered} \tag{2.19}$$

The dispersionless limit $\epsilon \to 0$ of (2.19) produces the semi-Hamiltonian system

$$a_t = (a^2 + 2b)_x, \quad b_t = -2c_x, \quad c_t = 2(ac)_x;$$

whereas the high frequency limit $\epsilon \to \infty$ gives

$$a_t = b_x, \quad b_t = (c_x - b a_x)_x, \quad c_t = \frac{1}{2}(b^2)_x.$$

This system is equivalent to WDVV by $a = f_{xx}$, $b = f_{xt}$ and $c$ is such that $c_x = f_{tt} + f_{xt}f_{xxx}$ and $c_t = f_{xt}f_{xxt}$. Both limit systems are bi-Hamiltonian, each equipped with a pair

of local Hamiltonian structures. The intermediate system (2.19) is also bi-Hamiltonian and possesses a pair of *quasi*-local Hamiltonian structures. In the standard scheme we thus have the following picture: System (2.4.2) is an integrable system of hydrodynamic type, the intermediate system (2.19) is one of its dispersive deformations (another one is the Yadjima-Oikawa system), WDVV equations then appear as another limit, $\epsilon \to \infty$. Furthermore, (2.19) is a system of Camassa-Holm type [4]. The Camassa-Holm equation is a non-evolutionary equation and the first example in the theory of integrable systems, which possesses infinitely many peakon solutions. Non-evolutionary systems can possess local Hamiltonian structures, but corresponding Hamiltonian densities are non-local or vice versa: if their Hamiltonian densities are local, corresponding Hamiltonian operators are non-local.

### 2.4.3 Exceptional cases: Bogojavlenskii's equations

Bogoyavlenskii's Breaking Soliton equations arise as a simple two-dimensional generalisation of well-known equations, by allowing the Lax pair to depend on an additional independent variable. The analogue of KdV, often called the Breaking Soliton equation (see [3]), written in its nonlocal form is

$$v_t - \frac{1}{2} v_y \partial_y^{-1} v_x - v v_x + \frac{\epsilon^2}{2} v_{xyy} = 0. \tag{2.20}$$

This equation is integrable, possesses a Lax pair and infinitely many commuting flows. What is remarkable is that its dispersionless limit

$$v_t + v v_x + u v_y = 0, \quad u_y = \frac{1}{2} v_x, \tag{2.21}$$

cannot be treated with the standard integrability test for multidimensional quasilinear systems, based on the method of hydrodynamic reductions, since the dispersion relation is degenerate — it reduces to two lines rather than being a conic [23]. Furthermore, the (2+1)-dimensional non-linear Schrödinger equation, which also appears in [3] as a breaking soliton generalisation of NLS

$$i\epsilon\, \psi_t + \epsilon^2 \psi_{xy} \pm 2\, \psi \partial_y^{-1} (|\psi|^2)_x \; = 0, \tag{2.22}$$

after an appropriate transformation (the so called Madelung transformation, see Section 5), gives rise in a dispersionless limit to

$$R^1_t + R^1 R^1_x + u R^1_y = 0, \quad R^2_t + R^2 R^2_x + u R^2_y = 0, \quad u_y = \frac{1}{2}(R^1 + R^2)_x. \tag{2.23}$$

Since both non-local systems (2.20) and (2.22) are integrable, their dispersionless limits (2.21) and (2.23) are also integrable (because they preserve infinitely many conservation

laws and higher commuting flows). So the question of how to understand their integrability, and more generally the integrability of the generalisation to $M$ components ($\kappa_i$ are constants)

$$R^i_t + R^i R^i_x + uR^i_y = 0, \quad u_y = \sum_{i=1}^{M} \kappa_i R^i_x, \quad i = 1, \dots, M,$$

arises naturally[1].

In [55] we consider an 'enveloping' four-dimensional integrable quasilinear system. We extract $M$ component three-dimensional hydrodynamic reductions, that cannot be further reduced, i.e. they would fail the hydrodynamic reductions [23] integrability test. However they preserve infinitely many conservation laws and commuting flows and are thus integrable. One of these reductions is the dispersionless limit of the breaking solution equation and contains higher commuting flows as well. More generally this means that the method of hydrodynamic reductions can be augmented by allowing high dimensional reductions (in this case (2+1)).

Moreover, our approach is universal, meaning that any $D$-dimensional quasilinear system from any hyper-Kähler hierarchy possesses $(D-1)$-dimensional hydrodynamic reductions.

Any multi-dimensional linearly degenerate equation of second order, which necessarily belongs to the hyper-Kähler hierarchy possesses infinitely many global solutions (see, for instance, [25]). However, the Breaking Soliton equations have no global solutions, thus the extraction of corresponding solutions from multi-dimensional linearly degenerate equations of second order remains an open problem.

[1]The first system (2.21) is linearisable by a point transformation of the dependent and independent variable $x = x(v, t)$. Such an approach, however, does not generalise to the multicomponent case.

# Chapter 3

# Mathematical physics via scientific computing

In this chapter we take a different viewpoint to study dispersive PDEs and phenomena related to them: we look at their solutions and we do this using numerical simulations. Many dispersive systems have known exact solutions. Integrable systems have entire infinite families of such exact solutions. However, as there are nonlinear systems, there is rarely a way to obtian all solutions in and exact form. Actually, some of the most interesting solutions, for example those related to shock waves are available only through numerical simulation.

We start our discussion with the presentation of the main tool used in this research: spectral collocation methods. We then present results relating to two families of multi-dimensional equations: Davey-Stewartson and Zakharov-Kuznesov type equations, where we discuss various phenomena, propose a number of conjectures and discuss limitations of the methods. At the end we very briefly describe other works related to special functions and transformations, that use the same method and are part f a toolbox for future research.

## 3.1 Spectral Differentiation

Here we very briefly present the numerical methods we use, that are based on spectral differentiation. The reader is invited to look at [63], which is a hands-on introduction to the subject. The aim of this techniques is to be able to differentiate in space with a high, and more importantly, controllable precision. A one-dimensional function $f(x)$ defined over some interval for $x$, is discretised into the set $f(x_i)$ for a grid $x_i$ for $i = 1, \dots N$. The inverse problem, how to recover a function $g(x)$ from the set $(g_i(x_i))$ cannot be solved exactly, but only up to a certain precision, that is we can construct an interpolant $\mathbf{g}$ such that $\mathbf{g}(x_i) = g_i$ and $|\mathbf{g}(x) - g(x)| \leq \epsilon_i$ for all $x$. We can then say we know the function $g(x)$ up to a said precision $\epsilon_i$. Most modern computers store real numbers in double-precision

floating point format. Algebraic operations in finite floating point format are not exact but precise to *machine precision*, also called machine epsilon $\epsilon_m$. For double float the machine precision $2^{-52} \approx 2.2210^{-16}$ i.e. we can track quantities that are up to 15 orders of magnitude different, which is sufficient for almost any application. An excellent practical interpolation method is such that the numerical precision is smaller than the machine epsilon $\epsilon \leq \epsilon_m$. What a "good" interpolation method is depends on the application e.g. in industrial machining six orders of magnitude are sufficient, in medical imaging even two may be enough. Spatial discretisation allows us to preform numerical algebraic operations as well as numerical differentiation and integration. As we are mostly concerned with PDEs we will concentrate on differentiation. Differentiation is a linear operation, thus it can be represented by the action of a *differentiation matrix* $D$, such that

$$f'_i = D^j_i f_j.$$

Each differentiation also has a finite precision namely the difference $|f'(x_i) - f'_i| \leq \epsilon_d$. The quantity $\epsilon_d$ is often called numerical precision (rather than e.g. differentiation precision), as it is most difficult to control and usually largest.

Depending on the functions and domains over which these functions are defined different spatial discretisation methods work best. In our work we mostly use two — Fourier discretisation to treat periodic functions as well as Schwartz functions (which, up to a finite precision, can be treated as periodic over a sufficiently large interval $[-L, L]$) and Chebyshev polynomials to treat functions on finite intervals or with algebraic decay.

Fourier discretisation is in a sense standard, and we will only briefly stop on it. We approximate a function with a Fourier series

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_i e^{ikx}.$$

The Fourier transform of a smooth, rapidly decreasing functions is rapidly decreasing, see the discussion in [63]. The same is true for periodic functions. Thus for analytic periodic functions for every $\epsilon$ there is $N$ such that $a_{\pm(N+i)} \leq \epsilon$. Thus we can truncate the infinite series and use a finite grid in both the $x$ and $k$ variables

$$f_i(x_i) = \frac{1}{2\pi} \sum_{k=-N+1}^{N} a_k e^{ikx_i}.$$

The coefficient $a_j$ can be obtained via the Fast Fourier Transformation (FFT), which we denote by $\mathcal{F}$, and the values $f_i$ can be obtained from the coefficients $a_i$ by its inverse, $\mathcal{F}^{-1}$. In the space of coefficients the Fourier differentiation matrix $D$ is diagonal following from the formula $\partial_x f(x) = \mathcal{F}^{-1} ik \mathcal{F}(f)$. One can construct Fourier differentiation matrices that work directly in the function space, however full matrix multiplication is of order $\mathcal{O}(N^2)$ operation, whereas FFT, thanks to the Cooley-Tukey algorithm is an $\mathcal{O}(N \log(N))$

operation, thus for sufficiently large $N$ it is much faster to do Fourier transforms back and forth.

For finite domains we approximate functions with a sum of Chebyshev polynomials $T_m(l) := \cos(m \arccos x)$, $l \in [-1, 1]$, i.e.,

$$f(l) \approx \sum_{m=0}^{N} a_m T_m(l). \tag{3.1}$$

The constants $a_m$ are determined via a collocation method: on the points

$$l_j = \cos\left(\frac{j\pi}{N_r}\right), \quad j = 0, \ldots, N$$

the equations (3.1) are imposed as equalities, i.e., for fixed $n$ we have

$$a(l_j) = \sum_{m=0}^{N} a_m T_m(l_j), \quad j = 0, \ldots, N$$

which uniquely determines the $a_n$. Because of the definition of the Chebychev polynomials, one has $T_m(l_j) = \cos(mj\pi/N_r)$. Thus the coefficients $a_n$ can be obtained via a *Fast Cosine Transformation* (FCT) which is related to the FFT, see e.g., the discussion in [63] and references therein.

To approximate derivatives via the ansatz (3.1), one uses $T_0'(l) = 0$, $T_1'(l) = 1$ and for $n \geq 1$ the identity

$$\frac{T_{n+1}'(l)}{n+1} - \frac{T_{n-1}'(l)}{n-1} = 2T_n(l)$$

which implies that the derivative of the $a_n(l)$ is approximated via the action of a *differentiation matrix* $D$ on the Chebychev coefficients $a_{nm}$

$$a_n'(l) \approx \sum_{m,\alpha=0}^{N_r} D_{m\alpha} a_{n\alpha} T_m(l)$$

The differentiation matrix is upper triangular and for even $N_r$ of the form

$$D = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 5 & \ldots & N_r - 1 & 0 \\ & & 4 & 0 & 8 & \ldots & & & 2N_r \\ & & & 6 & 0 & 10 & \ldots & 2(N_{r-1}) & 0 \\ & & & & 8 & 0 & 12 & \ldots & 2N_r \\ & & & & & \ddots & & & 0 \\ & & & & & & & & \vdots \\ & & & & & & & & 0 \\ & & & & & & & & 2N_r \end{pmatrix}.$$

In a similar way one can divide in the space of Chebychev coefficients by $l$ which is normally numerically a very delicate operation if $l$ can vanish on the considered interval as it does here. Using the identity

$$T_{n+1}(l) + T_{n-1}(l) = 2lT_n(x), \quad n = 1, 2, \ldots, \tag{3.2}$$

we can divide in coefficient space by $l \pm 1$. For given Chebyshev coefficients $a_{nm}$ we define the coefficients $\tilde{a}_{nm}$ via $\sum_{n=0}^{\infty} a_{nm}T_m(l) =: \sum_{n=0}^{\infty}(l \pm 1)\tilde{a}_{nm}T_m(l)$. This implies the action of a matrix $R$ in coefficient space (if $a_n(r)/r$ is bounded for $r \to 0$),

$$\frac{a_n(l)}{l} \approx \sum_{m,\alpha=0}^{N_r} R_{m\alpha}a_{n\alpha}T_m(l). \tag{3.3}$$

The matrix $R^{-1}$ has for even $N_r$ the form

$$R^{-1} = \begin{pmatrix} 1 & 1/2 & & & & \\ 1 & 1 & 1/2 & & & \\ & 1/2 & \ddots & \ddots & & \\ & & \ddots & & & \\ & & & & & 1/2 \\ & & & & 1/2 & 1 \end{pmatrix};$$

(the matrix $R$ is computed by inverting this matrix).

Spectral methods pose unique computational challenges. As the driving transformations FFT and FCT are global, this means that each operation uses the entire data array that describes a function $f$. In higher dimensions achieving high precision may require several thousand points in each spatial direction, which means that the arrays describing a complex function in double precision are from several GB to several tens of GB in size. Computations have to done locally on the data but it also has to be cross-communicated between the nodes. This is an enormous challenge for traditional computer systems, which even in the high-performance computing versions are *clusters* of computational nodes, that is, they are designed to separate a task in many parts and then each node works on its part independently. The emphasis is on computational power of each of the nodes in the cluster, and not on their inter-connectivity. This is an excellent hardware solution for e.g. FEM, FVM and other local methods, however global methods such as FFT cannot be efficiently scaled this way. To solve this problem we use graphical processing unit computing, taking advantage of the graphics processors to deploy mass-parallel computing and having very high memory transfer capability within each graphics card and good connection to the other cards. Commercial solutions have recently (2018) appeared under the name *dense computational systems* having several high-performance graphic cards connected on very

high speed connections, promising a new era in computing.

In the rest of the chapter we give brief descriptions of the Davey-Stewartson II equation, and the Zakharov-Kuznetsov equation and our results in this direction. The same approach has been used in a series of works on problems related to special functions and ODEs, namely the hypergeometric function, the Painlevé transcendents and Hilbert transforms and stability of Peregrine solutions to the NLS equation. All these propose methods for efficient, fast and precise computing of these functions and transformations on various domains or classes of functions. These results are intended as a toolbox for further research in PDEs. The reader is kindly invited to find more details in the respective articles [8, 44, 45, 46].

## 3.2 Davey-Stewartson

The Davey-Stewartson (DS) II family of equations is

$$\begin{aligned} i\Psi_t + \Psi_{xx} - \Psi_{yy} + 2\rho\left(\beta\Phi + |\Psi|^2\right)\Psi &= 0, \\ \Phi_{xx} + \Phi_{yy} + 2\left|\Psi\right|^2_{xx} &= 0, \end{aligned} \tag{3.4}$$

where $\beta$ is a positive constant, indices denote partial derivatives, $\rho$ takes values $\pm 1$, and where $\Phi$ denotes a mean field. The systems (3.4) are of considerable importance in applications since they are a simplification of the Benney-Roskes [2], or Zakharov-Rubenchik [67] systems being 'universal' models for the description of the interaction of short and long waves. They first appeared in the context of water waves [10, 11, 1, 49] in the so-called modulational (Schrödinger) regime, i.e., in the study of the modulation of plane waves. In [6, 7] is was shown, via diffractive geometric optics, that DS systems provide good approximate solutions to general quadratic hyperbolic systems. Furthermore, the DS systems appear in numerous physical contexts as ferromagnetism [48], plasma physics [53], and nonlinear optics [54]. The Davey-Stewartson systems can also be viewed as the two-dimensional version of the Zakharov-Schulman systems (see [68, 69]. For more details on DS and its applications the reader is referred to [39, 38] where an abundance of references can be found.

The Davey-Stewartson II equation is completely integrable, possessing a Lax-pair

$$\psi_x + i\sigma_3\psi_y = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}\psi, \tag{3.5}$$

$$\psi_t = \begin{pmatrix} i\overline{\partial}^{-1}\partial\left(|q|^2\right)/2 & -i\partial q \\ i\overline{\partial}\overline{q} & -i\overline{\partial}^{-1}\partial\left(|q|^2\right)/2 \end{pmatrix}\psi - \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}\psi_y + i\sigma_3\psi_{yy}\ . \tag{3.6}$$

The equation (3.5) is referred to, by analogy with previous 1-dimensional cases, as the

spectral problem associated to the DS-II equation. It can equivalently be written as

$$\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi = \frac{1}{2} \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \psi \ . \tag{3.7}$$

The *direct problem* is then to seek a vector-valued solution $\psi = \psi(z,k) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with the following asymptotic behavior as $|z| \to \infty$:

$$\lim_{|z|\to\infty} \psi_1 e^{-kz} = 1, \tag{3.8}$$

$$\lim_{|z|\to\infty} \psi_2 e^{-\overline{k}\overline{z}} = 0 \ . \tag{3.9}$$

The quantity $\psi$ is referred to as a CGO solution. The *reflection coefficient*, $r(k)$, is encoded in the sub-leading term in the asymptotic expansion of $\psi$, via

$$\psi_2 e^{-\overline{k}\overline{z}} = \frac{r(k)}{\overline{z}} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \ . \tag{3.10}$$

Now, if $q = q(x,y,t,\epsilon)$ evolves according to the DSII equation, then the reflection coefficient evolves according to

$$r = r^\epsilon(k,t) = r^\epsilon(k,0) e^{\frac{-it}{4\epsilon}\left(k^2+\overline{k}^2\right)} \ .$$

In this sense, the mapping from $q$ to $r$ linearises the DSII flow. More amazingly, the inverse problem of reconstructing the potential $q(x,y,t,\epsilon)$ from the reflection coefficient $r(k,t)$ is also a D-bar problem, only in the complex variable $k$. Indeed, setting

$$\Phi_1 = \Phi_1^\epsilon(k;z,t) := \mathrm{e}^{-kz/\epsilon}\psi_1 \quad \text{and} \quad \Phi_2 = \Phi_2^\epsilon(k;z,t) := \mathrm{e}^{-\overline{k}\overline{z}/\epsilon}\psi_2 \ , \tag{3.11}$$

it turns out that one has, for each $z \in \mathbb{C}$,

$$\epsilon\overline{\partial}_k \Phi_1 = \tfrac{1}{2} e^{(\overline{k}\overline{z}-kz)/\epsilon} \overline{r^\epsilon(k,t)} \Phi_2, \qquad \epsilon\partial_k \Phi_2 = \tfrac{1}{2} e^{-(\overline{k}\overline{z}-kz)/\epsilon} r^\epsilon(k,t) \Phi_1 \tag{3.12}$$

where,

$$\overline{\partial}_k := \frac{1}{2}\left(\frac{\partial}{\partial k_1} + \mathrm{i}\frac{\partial}{\partial k_2}\right), \qquad \partial_k := \frac{1}{2}\left(\frac{\partial}{\partial k_1} - \mathrm{i}\frac{\partial}{\partial k_2}\right),$$

and the asymptotic conditions

$$\lim_{|k|\to\infty} \Phi_1^\epsilon(k;z,t) = 1 \quad \text{and} \quad \lim_{|k|\to\infty} \Phi_2^\epsilon(k;z,t) = 0. \tag{3.13}$$

The functions $\Phi_1$ and $\Phi_2$, being uniquely determined by the above elliptic system (3.12) and boundary conditions (3.13), yield the potential $q(x,y,t,\epsilon)$ through the asymptotic behavior as $|k| \to \infty$:

$$\Phi_2 = \frac{\overline{q(x,y,t,\epsilon)}}{2k} + \mathcal{O}\left(|k|^{-2}\right) \ .$$

DSII has soliton solutions called lumps, which have algebraic decay towards infinity. DSI supports so called "dromions", which have non-trivial behaviour at infinity.

### 3.2.1 Results

The Davey-Stewartson equation can be thought of as a non-local NLS type equation. The general theory of such equations [59] suggests that under certain conditions this type of equations can experience *blow-up*, the production of a singularity from smooth initial data. DS II is remarkable as there is even an explicit solution, due to Ozawa that becomes singular at one point in space and time. The Ozawa solution has an exact blow-up by applying a pseudo-conformal transformation $\tilde{\psi}(x,y,t) = \exp\left(\frac{i(x^2-y^2)}{4t}\right)\psi\left(\frac{x}{t},\frac{y}{t},\frac{1}{t}\right)$ on a standard lump. Let $ab<1$ and $T=-a/b$. Let $u(x,y,t)$ is the function

$$u(x,y,t) = exp(i\frac{b}{4(a+bt)}(x^2-y^2))\frac{v(X,Y)}{a+bt},$$

where

$$v(X,Y) = \frac{2}{1+X^2+Y^2}, \quad X = \frac{x}{a+bt}, \quad Y = \frac{y}{a+bt}.$$

Then $u$ is a solution to DS II with $||u(x,y,t)||_2 = ||v(X,Y)||_2 = 2\sqrt{\pi}$. Furthermore, $|u(t)|^2 \to 2\pi\delta$ in $\mathcal{S}'$ when $t \to T$

$$|\partial_x\psi(t,)|_2^2 \propto (t^*-t)^{-2}, \quad |\psi(t,)|_\infty \propto \psi(t^*-t)^{-1}.$$

This seems at odd with the classical theory result (see again [59] and also [51]), that via dynamical rescaling

$$X = x/L(t), \quad Y = y/L(t), \quad \tau = \int_0^t \frac{dt'}{L^2(t')} \quad \Psi(X,Y,\tau) = L(t)\psi(x,y,t).$$

$$|\psi|_2 = |\Psi|_2$$

paved the way for studying blow-up in focusing NLS analytically :in cases with a generic blow-up in one point, the blow-up of the quintic NLS solution is self-similar and follows the 'dynamical rescaling' $L(t) \propto \sqrt{\frac{t^*-t}{\ln|\ln(t^*-t)|}}$ and thus

$$|\partial_x\psi(t,)|_2^2 \propto (t^*-t)^{-1}, \quad |\psi(t,)|_\infty \propto \psi(t^*-t)^{-1/2}.$$

In [42] we study various initial data and propose the following

**Conjecture 6.** *Consider initial data $\psi_0 \in C^\infty(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ for the focusing DS II equation*

$$i\partial_t\psi + \Box\psi + 2[(\Delta^{-1}\Box)|\psi|^2]\psi = 0$$

*with a single global maximum of $|\psi_0|$ such that the solution to DS II has a blow-up in finite time. Then the blow-up is self-similar with*

$$X = \frac{x}{L(t)}, \quad Y = \frac{y}{L(t)}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')}, \quad \Psi(X,Y,\tau) = L(t)\psi(x,y,t).$$

*with a scaling factor $L(t) \propto t^* - t$, and the blow-up profile given by the lump, i.e.,*

$$\psi(x, y, t) = \frac{P(X, Y)}{L(t)} + \tilde{\psi}, \quad P(X, Y) = \frac{2}{1 + X^2 + Y^2},$$

*where $\tilde{\psi}$ is bounded for all $t$.*

That is, the Ozawa-type blow up is generic and localised data of sufficient mass will develop a singularity in finite time, and the asymptotic profile will be that of a scaled lump. In [33] we develop a high precision method for Fourier discretisation of DSII by using an analytic regularisation (within the machine epsilon) of the nonlocal term in Fourier space, thus producing a much more robust computational tool.

In the series of articles [34, 43, 40, 41] we consider the scattering problem for DSII given by d-bar equation and in particular its complex geometric optics (CGO) solutions for Schwartz potentials as well as potentials with compact support. We propose methods exhibiting spectral convergence to compute said CGO solutions as well as the reflection coefficient. The presence of rapidly oscillating terms in (3.12) makes numerical computation for high values of the spectral parameter $k$ difficult, thus in the last two of the mentioned articles we complement the numerical computation for low and intermediate values of $k$ with exact asymptotic formulas for large $k$, thus constructing a hybrid computational method. The main *computational result* are the estimates

$$\phi_2^1 = \frac{1}{2k} e^{i|k|\Re(\cdot\overline{\omega})} 1_\Omega + \mathcal{O}(1) h^{3/2} (\ln(1/h))^{1/2} \text{ in } \langle\cdot\rangle^\epsilon L^2,$$

and

$$\phi_1^1 = \frac{h}{4k} E(1_\Omega) + \mathcal{O}(1) h^{3/2} (\ln(1/h))^{1/2} \text{ in } \langle\cdot\rangle^\epsilon L^2.$$

This allows us to estimate the reflection coefficient as well.

D-bar equations find various other applications, one of the most remarkable of which is in solving what is called the Calderón problem, namely how to recover the conductivity $\gamma$ in a given domain $\Omega$ from measurements on the boundary $\partial\Omega$ (for a thorough review and references see [65]). The conductivity equation in 2d is

$$\nabla \cdot (\gamma(x_1, x_2) \nabla u(x_1, x_2)) = 0, \quad (x_1, x_2) \in \Omega.$$

The voltage $f$ on $\partial\Omega$: Dirichlet condition $u(x_1, x_2)|_{\partial\Omega} = f(x_1, x_2)$, the current density $g$ on $\partial\Omega$ is a Neumann condition $\gamma(x_1, x_2)\partial_\nu u(x_1, x_2)|_{\partial\Omega} = g(x_1, x_2)$. The problem is then how to recover the conductivity in $\gamma(x, y)$ for the entire domain from the Dirichlet-to-Neumann mapping $\Lambda_\gamma$ (the Neumann data function $g$ from given Dirichlet data $f$). We hope to be able to apply the proposed computational methods in this direction as well.

## 3.3 Zakharov-Kuznetsov

Integrable structure provide aid, but our methods are not necessary restricted to them. The generalized Zakharov-Kuznetsov (ZK) equation

$$u_t + (\Delta u + u^p)_x = 0, \quad p = 2, 3, 4. \tag{3.14}$$

This equation can be considered as a multi dimensional generalisation of the Korteweg-de Vries (KdV) equation. Originally the equation was proposed by Zakharov and Kuznetsov in the 3D setting, see [66] to model waves in magnetized plazmas, the first rigorous derivation was done by Lannes, Linares and Saut in [47] from the Euler-Poisson system. In 2D the quadratic ($p = 2$) ZK equation governs, for example, weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [52]. In [50] this equation appears as the amplitude equation for two-dimensional long waves on the free surface of a thin film flowing down a vertical plane with moderate values of the fluid surface tension and large viscosity.

From the local theory it follows that solutions to the ZK equation have a maximal forward lifespan $[0, T)$ with either $T = +\infty$ or $T < +\infty$. In the later case in the 2D setting one has $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \nearrow \infty$ as $t \to T$, though the unbounded growth of the gradient might also happen in infinite time.

During their existence, solutions to ZK have several conserved quantities, relevant to this work is the $L^2$ norm (or mass), and the energy (or Hamiltonian):

$$M[u(t)] = \int_{\mathbb{R}^2} u^2(t) = M[u(0)],$$

$$E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^2} \left(u_x^2(t) + u_y^2(t)\right) - \frac{1}{p+1}\int_{\mathbb{R}^2} u^{p+1}(t) = E[u(0)]. \tag{3.15}$$

Unlike the 1D KdV or modified KdV, the ZK equation is not integrable for any power $p$.

An important symmetry in the evolution equations is *the scaling invariance*, which states that an appropriately rescaled version of the original solution is also a solution of the equation. For the equation (3.14) it is

$$u_\lambda(\mathbf{r}, t) = \lambda^{\frac{d}{p-1}} u(\lambda x, \lambda y, \lambda^3 t). \tag{3.16}$$

This symmetry makes invariant the Sobolev norm $\dot{H}^s$ with $s = 1 - \frac{d}{p-1}$, since $\|u_\lambda\|_{\dot{H}^s} = \lambda^{\frac{d}{p-1}+s-1}\|u\|_{\dot{H}^s}$. Moreover, the index $s$ gives rise to the critical-type classification of (3.14): when $s < 0$, or $p < d + 1$, the equation (3.14) is called the $L^2$-subcritical equation (in this paper a representative of this case is $p = 2$); if $p > d + 1$, or $s > 0$, the equation is $L^2$-supercritical (we use $p = 4$), and with $p = d + 1$, or $s = 0$, it is $L^2$-critical. This classification is important in the study of long time behaviour of solutions. For that we

need the notion of solitons, as solutions in general position usually resolve to solitons and radiation which is clearly distinct.

ZK equations have a family of localized travelling waves (or solitary waves, often referred to as solitons), which travel only in $x$ direction

$$u(x, \bar{r}, t) = Q(x - ct, \bar{r}) \tag{3.17}$$

where $\bar{r} = y$ in the 2D case and $\bar{r} = (y, z)$ in the 3D case and $Q$ satisfies

$$-cQ + \Delta Q + Q^p = 0; \tag{3.18}$$

and defining the ground state solution (i.e., the unique radial positive $H^1$ solution vanishing at infinity, for which the existence, uniqueness and various other properties are well-known, see for example, [61]). We note that $Q \in C^\infty(\mathbb{R}^2)$, $\partial_r Q(r) < 0$ for any $r = |(x, y)| > 0$, and that $Q$ has exponential decay $|\partial^\alpha Q(x, y)| \leq c_\alpha\, e^{-r}$ for any multi-index $\alpha$ and any $(x, y) \in \mathbb{R}^2$. The solitons $Q_c(x, y)$ are related to the soliton $Q_1(x, y) =: Q(x, y)$ for $c > 0$ via

$$Q_c(x, y) = c^{\frac{1}{p-1}}\, Q(\sqrt{c}\, x, \sqrt{c}\, y), \tag{3.19}$$

thus, it suffices to consider $c = 1$.

### 3.3.1 Results

In [36] we study the long term behaviour as well as critical phenomena for the 2D ZK equation in the subcritical, critical and supercritical case. The reader is invited to look at the aforementioned article, which contains a number of plots, giving flesh and blood to the following conjectures:

**Conjecture 7** ($L^2$-subcritical case)**.** *Consider the subcritical 2D ZK equation, in particular, when $p = 2$ in* (3.14)*.*

1. *The soliton solutions* (3.17)-(3.18)-(3.19) *are orbitally and asymptotically stable.*
2. *Solutions of* (3.14) *with general sufficiently localized initial data and of sufficient smoothness decompose as $t \to \infty$ into solitons and radiation.*

**Conjecture 8** ($L^2$-critical case)**.** *Consider the critical 2D ZK equation* (3.14) *with $p = 3$.*

1. *If $u_0 \in \mathcal{S}(\mathbb{R}^2)$ is such that $\|u_0\|_2 < \|Q\|_2$, then the solution $u(t)$ to* (3.14) *is dispersed.*
2. *If $u_0 \in \mathcal{S}(\mathbb{R}^2)$ is sufficiently localized and such that $\|u_0\|_2 > \|Q\|_2$, then the solution blows up in finite time $t = t^*$ and such that as $t \to t^*$*

$$u(x, y, t) - \frac{1}{L(t)}\, Q\left(\frac{x - x_m(t)}{L(t)}, \frac{y - y_m(t)}{L(t)}\right) \to \tilde{u} \in L^2, \tag{3.20}$$

*with*

$$\|u_x(t)\|_2 \sim \frac{1}{L(t)}, \;\; L(t) \sim \sqrt{t^* - t}, \quad \textit{and} \quad x_m(t) \sim \frac{1}{t^* - t}, \;\; y_m(t) \to y^* < \infty. \tag{3.21}$$

**Conjecture 9** ($L^2$-supercritical case)**.** *Consider the supercritical 2D ZK equation, in particular, when $p = 4$ in* (3.14)*. Let $u_0 \in \mathcal{S}(\mathbb{R}^2)$ be of sufficiently large mass and energy and of some localization. Then ZK evolution $u(t)$ blows up in finite time $t^*$ and finite location $(x^*, y^*)$, i.e., the blow-up core resembles a self-similar structure with*

$$u(x,y,t) - \frac{1}{L^{\frac{2}{p-1}}(t)} P\left(\frac{x - x_m(t)}{L(t)}, \frac{y - y_m(t)}{L(t)}\right) \to \tilde{u} \in L^2, \tag{3.22}$$

*where $P(x, y)$ is a localized solution to* (3.14) *(which is conjectured to exist),*

$$x_m(t) \to x^*, \quad y_m(t) \to y^*,$$

*and*

$$\|u_x(t)\|_2 \sim \frac{1}{L^{\frac{2}{p-1}}(t)} \quad \text{with} \quad L(t) \sim (t^* - t)^{1/3} \quad \text{as} \quad t \to t^*. \tag{3.23}$$

The 3D case is considered in [37]. As 3D spatial discretisation requires significant resources and blow-up tracking requires significant precision we only study the subcritical case. The critical and supercritical are part of ongoing research. We propose the following conjecture:

**Conjecture 10.** *Consider the 3D ZK equation* (3.14)*.*

1. *The soliton solutions* (3.17)-(3.18) *are orbitally and asymptotically stable.*
2. *Solutions of* (3.14) *with general sufficiently localized initial data and of sufficient smoothness decompose as $t \to \infty$ into solitons and radiation.*

# Chapter 4

# Conclusion

We have presented two viewpoints on nonlinear partial differential equations that try to understand some of the structures behind these equations and give a closer understanding of the phenomena that occur. This work does not even pretend to be a beginning of a comprehensive study of the subject.Large swathes of knowledge, important aspects and approaches have been omitted or barely touched, for example classical PDE analysis, inverse problems and relation to physical applications and experiments, to name a few.

Many challenging questions remain. It is our conviction that approaches combining analytical, numerical, technological knowledge have the best chance of surmounting the difficulties along the way. This is the spirit this work aims to convey.

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