# International Master in Mathematical Physics

## – Math4Phys –

Guide 2024-25

Contents

## 1. Presentation

The Master in Mathematical Physics "Math4Phys" is a master course of study of the Department of Mathematics of the *Université de Bourgogne* (uB), which takes place at the *Institut de Mathématiques de Bourgogne* (IMB) in Dijon.

The main aim of the Master is to provide advanced lectures on the mathematical methods of modern theoretical physics in the framework of a mathematical curriculum.

Such an offer exists in France only in Dijon as the Mathematical Physics group of the IMB provides a unique environment for a program requiring a double competence in Mathematics and Physics.

The Mathematical Physics group of the IMB laboratory in Dijon is a unique research team in France with the ability to provide advanced lectures on the mathematical problems of modern physics.

## 2. Contacts

Official web page: `math.u-bourgogne.fr/math4phys`

General enquiries: `math4phys@u-bourgogne.fr`

Address:
Institut de Mathématiques de Bourgogne, UMR CNRS 5584,
9 avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France

Secretariat:
Mylène MONGIN – `secretariat.maths@u-bourgogne.fr`

International Masters coordinator (International students support):
Eloïse ROUSSEL – `eloise.roussel@u-bourgogne.fr`

Coordinators:
Guido CARLET – M1 – office A415 – `guido.carlet@u-bourgogne.fr`
José Luis JARAMILLO – M1 – office A323 – `jose-luis.jaramillo@u-bourgogne.fr`
Nikolai KITANINE – M2 – office A405 – `nikolai.kitanine@u-bourgogne.fr`

## 3. Application, enrolment, and other useful information

For detailed information in English about the application and enrolment procedures you can refer to the webpage:
`en.u-bourgogne.fr/admission/degree-seeking-students.html`

Applications are submitted uniquely via the eCandidat website:
`ecandidat.u-bourgogne.fr/`

## 4. Scholarships

Each year the Master awards several scholarships of about 550 euros per month for up to 9 months to the most deserving students. The number of the available scholarships depends on the secured funding. All students accepted in M1 and M2 are automatically considered for the scholarships, which will be attributed solely on the basis of academic merit. Scholarship awardees are required to confirm their participation within two weeks of the offer of the scholarship.

## 5. Calendar 2024/25

| | |
|---|---|
| Applications begin | 12/2/2024 |
| Applications end | 24/5 |
| First meeting M1 | 16/9 |
| First meeting M2 | 16/9 |
| *Toussaint* * | 26/10 – 4/11 |
| Courses end | 20/12 |
| *Noël* | 21/12 – 6/1 |
| Exams | 6/1 – 10/1 |
| Courses begin | 13/1/2025 |
| *Hiver* | 22/2 – 3/3 |
| *Printemps* | 19/4 – 5/5 |
| Courses end | 9/5 |
| Dissertation deadline | 20/5 |
| Exams, 1st session | 12/5 – 16/5 |
| Exams, 2nd session | 16/6 – 27/6 |

* In *italics* the teaching breaks.

## 6. Courses M1

We propose nine main courses plus a language course and a dissertation. Each course (apart from the language course) consists of 22 hours of lectures (CM) and 22 hours of exercise classes (TD). For French-speaking students the course FLE will be replaced by a course of English.

| | CM+TD (hrs) | ECTS |
|---|---|---|
| Groups and representations | 22+22 | 7 |
| Differential geometry | 22+22 | 7 |
| Functional analysis | 22+22 | 7 |
| Differential equations in the complex domain | 22+22 | 7 |
| FLE (or English) | 0+20 | 2 |
| Mathematical methods of classical mechanics* | 22+22 | 6 |
| Quantum mechanics for mathematicians* | 22+22 | 6 |
| Partial differential equations* | 22+22 | 6 |
| Computational methods in mathematical physics* | 22+22 | 6 |
| Statistical mechanics and stochastic processes* | 22+22 | 6 |
| Dissertation | | 6 |

* The students will have to choose 4 courses out of the 5 options available.

The schedule of the courses will be accessible via the ENT website of the *Université de Bourgogne* at the address: `ent.u-bourgogne.fr`

All courses are taught by members of the IMB in Dijon, unless otherwise noted. Other affiliations include the *Laboratoire Interdisciplinaire Carnot de Bourgogne* (ICB).

*Groups and representations* (P. Schauenburg, S. Carrozza)

Notion of a group representation. Development of the structure theory for complex representations of finite groups: Theorems of Maschke and Schur. Tensor products and duality. Character theory. Induced representations. Some outlook beyond finite groups as time permits.

*Differential geometry* (R. Uribe-Vargas, M. Fairon)

Differentiable manifolds. Vector fields and flow-box theorem. Differential forms and Stokes' theorem. Tensors and vector bundles. Riemannian manifolds and connections. Geometry of gauge fields.

*Functional analysis* (G. Dito, T. Chambrion)

1. Basic notion on topology. Compacts spaces, bounded linear maps on normed spaces. Compact and complete spaces. 2. Hilbert spaces. Projection theorem. Hilbertian orthonormal basis. Riesz representation theorem. Applications to Fourier series. Fejér theorem. 3. Linear operators on a Hilbert space. General theory of operators. Adjoint of an operator. Closed, self-adjoint, symmetric, unitary, normal operators. Application to differential operators. 4. Spectral theory. Basic concepts: point, continuous, and residual spectra. Resolvent operator. 5. Spectral theorem for compact operators. Applications to integral equations. Spectral theorem for bounded self-adjoint operators.

*Differential equations in the complex domain* (G. Carlet, N. Kitanine)

ODEs in the real domain: elementary methods, existence and uniqueness theorems (Cauchy-Peano, Picard-Lindelöf), linear systems. ODEs in the complex domain: scalar equations of the first and second order, systems, Riccati equation; complex linear systems, monodromy and singularities, Fuchsian systems, Bessel equation, hypergeometric equation; irregular singularities, Stokes matrices.

*Mathematical methods of classical mechanics* (J. L. Jaramillo, B. Raffaelli)

Lagrangian and Hamiltonian formalisms. Hamiltonian systems on symplectic manifolds. Variational principle and Hamilton-Jacobi equations. Poisson manifolds. Symmetries and momentum map.

*Quantum mechanics for mathematicians* (N. Kitanine, S. Leurent)

1. Introduction: 1.1 Observables in classical mechanics 1.2. Finite dimensional model of quantum mechanics 2. Basic principles of quantum mechanics: 2.1 States and observables in quantum mechanics 2.2 Quantum entanglement 2.3 Heisenberg uncertainty principle 2.4 Coordinate and momentum representations 2.6 Quantum dynamics: Schrödinger and Heisenberg pictures 2.5 Schrödinger equation 2.6 Classical limit 3. Quantum mechanics in one dimension: 3.1 Harmonic oscillator. Creation and annihilation operators 3.2 Scattering problem in one dimension 4. Quantum mechanics in 3D: 4.1 Free particle 4.2 Rotation group and angular momentum 4.3 Hydrogen atom 4.4 Spin 5. Multi-particle quantum systems, introduction.

*Partial differential equations* (J. Lampart – ICB)

This course is an introduction to linear partial differential equations and relevant concepts from functional analysis. Tempered distributions; Fourier transform; Sobolev spaces; PDE with constant coefficients. Hilbert spaces; Bounded operators, linear functionals; Lax-Milgram Theorem and elliptic equations. Unbounded Operators; Self-adjoint and maximal dissipative operators; Evolution equations and the Hille-Yosida Theorem. Applications to the heat, wave and Schrödinger equations with potentials and variable coefficients.

*Computational methods in mathematical physics* (N. Stoilov)

Interpolation and/or Linear systems. Numerical integration (classical rules, Gaussian quadrature rules). Fourier approximation. Numerical methods for solving ODE and PDE.

*Statistical mechanics and stochastic processes* (J. L. Jaramillo, S. Fang)

*Statistical mechanics*: 1. Thermodynamics formalism. 2. Classical Statistical Mechanics and Ergodic Hypothesis. 3. Statistical Ensembles (and Large Deviation Theory). 4. Phase transitions, Ising model, renormalization group. 5. Non-equilibrium statistical mechanics: Onsager reciprocity and fluctuation-dissipation theorem. *Stochastic processes*: 6. Probability concepts. 7. Brownian motion and diffusion. 8. Ito calculus and stochastic differential equations. 9. The Fokker-Planck equation. 10. Detailed balance: Onsager relations and fluctuation-dissipation theorem (revisited).

## 7. Courses M2

There are eight main courses plus a course of languages and a dissertation. Three courses will be on the specific theme of the year. For French-speaking students the course FLE will be replaced by a course of English.

| | CM+TD (hrs) | ECTS |
|---|---|---|
| Lie groups and Lie algebras | 18+18 | 7 |
| Mathematical methods of quantum field theory | 18+18 | 7 |
| Riemann surfaces and integrable systems | 18+18 | 7 |
| An introduction to low-dimensional topology* | 15+15 | 7 |
| FLE (or English) | 0+20 | 2 |
| Path integral approach in QFT | 15+15 | 5 |
| General relativity | 15+15 | 5 |
| Tensor categories and quantum groups* | 15+15 | 5 |
| An introduction to topological quantum field theories* | 15+15 | 5 |
| Dissertation | | 10 |

* Thematic courses 2024-25.

The schedule of the courses will be accessible via the ENT website of the *Université de Bourgogne* at the address: `ent.u-bourgogne.fr`

All courses are taught by members of the IMB.

### 7.1. **Curricular courses.**

*Lie groups and Lie algebras* (G. Dito)

1. Lie algebras: Basic definitions. Ideals and Lie subalgebras. Lie theorems. Real and complex forms. Universal enveloping algebra. Poincaré-Birkhoff-Witt theorem. Campbell-Hausdorff formula. 2. Structure of Lie algebras: Solvable, nilpotent and semisimple Lie algebras. Killing form. Lie and Engel theorems. Cartan criterion. Jordan decomposition. 3. Semisimple Lie algebras: Cartan subalgebra. Root system. Dynkin diagram. Classification of simple Lie algebras. Finite dimensional representations of sl(2).

*Mathematical methods of quantum field theory* (N. Kitanine)

1. Introduction, necessary background in mathematics and physics: distribution theory, functional

analysis (spectral theorem), Lagrangian and Hamiltonian mechanics, basic principles of quantum mechanics, special relativity (Lorentz group and corresponding Lie algebra). 2. Classical field theory: Lagrangian formulation, conservation laws, Noether theorem for fields, examples (Klein-Gordon equation, sine-Gordon equation, non-linear Schrodinger equation, Maxwell equations etc.). Hamiltonian formulation, Poisson brackets. 3. Canonical quantisation of the free bosonic field: Weyl algebra, Fock space for bosons, lattice bosons, finite volume, infinite volume limit. Free bosons in 3+1 d. 4. Spinor representation of the Lorentz group and Dirac equation. Canonical quantization: Clifford algebra, Fock space for fermions: lattice fermions, finite volume, infinite volume limit. Free fermions in 3+1 d. 5. Introduction to the interacting field theories: Interaction picture, Dyson formula, Wick theorem, Feynman diagrams. 6. (if there is time) Introduction to quantum integrability.

*Riemann surfaces and integrable systems* (G. Carlet)
Definition of Riemann surface and basic examples, plane algebraic curves, hyperelliptic curves, holomorphic coverings, fundamental group, Riemann-Hurwitz theorem, homology groups, sheaves, Cech and sheaf cohomology, meromorphic functions, abelian differentials, integration theorems, divisors, Abel-Jacobi map, Abel theorem, Riemann-Roch theorem, Serre duality.

*Path integral approach in QFT* (T. Kimura)
Lagrangian formalism and symmetry. Path integral formalism. Interacting fields and perturbation theory. Loop correction and renormalization. Quantization of non-Abelian gauge theory. Spontaneous symmetry breaking.

*General relativity* (S. Carrozza, B. Raffaelli)
The bulk of the course will focus on basic aspects of General Relativity. Outline: 1. Minkowski spacetime: special relativity, proper time, metric, causal structure and conformal compactification, energy-momentum tensor. 2. Geometry of curved spacetimes: manifolds, tensor fields, Lie derivative, covariant derivatives, Levi-Civita connection, curvature, geodesics. 3. Einstein's field equations: (heuristic) derivation, Lovelock theorem, linearized gravity, Newtonian limit. 4. Particular solutions of Einstein's equations: Schwarzschild black hole, cosmological solutions, and their singularities. Time-permitting, we will explore the concept of spacetime singularity in more depth, by reviewing more advanced notions such as: congruences of curves, the Raychaudhuri equation, energy conditions, and singularity theorems.

### 7.2. **Thematic courses 2024-25.**

Each year the master proposes thematic courses on a specific theme. For the year 2024-25 the theme is "Introduction to quantum topology". We propose three courses. The first course gives the mathematical background for the object of study in quantum topology, i.e. low-dimensional manifolds (mainly 3-manifolds, including knots and links). The second course is an introduction to quantum algebra, and presents the algebraic structures underlying the construction of quantum invariants. The last and third course will provide examples of quantum invariants of 3-manifolds, and will consider their extensions to TQFTs ("Topological Quantum Field Theories").

*An introduction to low-dimensional topology* (G. Massuyeau)
Low-dimensional topology is characterized by its object of study, namely manifolds of "low" dimensions (no more than 4). In this introductory course, we will start with a short review of the classification of surfaces. Next, we will give an introduction to knots, links and tangles, which constitute special classes of 3-manifolds. In particular, we will define and study braid groups. Finally, we will see how to present arbitrary 3-manifolds via handle decompositions and surgery techniques. Although our approach will

be mostly based on the study of examples, we shall also need some basic tools of algebraic topology and/or differential topology.

*Tensor categories and quantum groups* (P. Schauenberg)

Tensor categories and quantum groups are at the heart of a plethora of applications of algebraic methods to mathematical physics and topology often summarized under the moniker quantum algebra. Key words include topological quantum field theory, conformal field theory, the theory of quantum computation, integrable systems, deformation quantization, and noncommutative geometry. The lecture will give an introduction to the basic notions in the theory of tensor categories and important additional structures such as braidings and duality, an introduction to quantum groups (Hopf algebras) whose representation theory gives important examples of such structures, and discuss important constructions, notably the Drinfeld center/double and quantized enveloping algebras.

*An introduction to topological quantum field theories* (L. Woike, R. Detcherry)

The notion of a topological quantum field theory axiomatizes a certain type of quantum field theory for which the quantities associated to a region of spacetime only depend on the 'shape' of the spacetime. In addition to their physical origin, topological quantum field theories are also a subject of purely mathematical interest, and this course will be focused on the mathematical aspects. In the first half of the course, we will introduce the general axiomatics of topological quantum field theories compactly as symmetric monoidal functors from the cobordism category to the category of vector spaces. Then we will focus on the classification of two-dimensional topological field theories by commutative Frobenius algebras. In the second half, we build a class of examples of three-dimensional topological quantum field theories, the so-called Dijkgraaf–Witten theories, via the finite path integral quantization of a discrete gauge theory. As an outlook, we will discuss aspects of the Reshetikhin–Turaev construction of topological field theories from modular fusion categories.

### 7.3. **Thematic courses 2025-26.**

For the year 2025-26 the theme will be "Algebraic geometry".

*Introduction to algebraic geometry* (L. Moser-Jauslin)

This course will concentrate on the study of affine and projective complex varieties. In the first part, we will discuss the correspondence between varieties and their rings of functions. We will prove Hilbert's Nullstellensatz, study its consequences and use commutative algebra to study varieties. We will introduce the Zariski topology, and study its relationship to the usual topology of complex varieties. After considering affine varieties, we will then concentrate on the case of projective curves. We will consider geometric and topological properties of algebraic varieties. We will start with elementary properties, such as smoothness and tangent spaces, and then later consider more involved notions such as intersections and also the genus of smooth projective curves.

*Cohomology of algebraic varieties* (J. Nagel)

The aim of this course is to discuss several topics related to the cohomology of complex projective algebraic varieties. We start out with singular cohomology and de Rham cohomology, and then move on to discuss the implications of a complex structure on the cohomology: Dolbeault cohomology and the Hodge decomposition theorem. In the last part of the course we study applications to mirror symmetry. To this end, we need orbifolds (varieties that locally look like the quotient of a smooth variety by a finite group action) and Chen-Ruan cohomology, a cohomology theory that reflects the orbifold structure.

*Introduction to Hilbert schemes and moduli spaces* (D. Faenzi)

This course is intended as an introduction to moduli spaces in algebraic geometry, through the guiding example of Hilbert schemes of points on a surface. 1. Moduli functors, Hilbert schemes. We will introduce the functorial approach to moduli problems, specify it to the case of Hilbert schemes and review some basic results on Hilbert schemes of $n$ points on a smooth algebraic surface. We will see that this is a smooth and irreducible variety of dimension $2n$, that admits a natural birational morphism to the symmetric $n$-th power of the surface, which is actually a resolution of singularities. 2. Higher rank moduli spaces. This part is devoted to the study of moduli spaces of framed sheaves on the projective plane and the associated ADHM data. This relies on the concept of moduli space of vector bundles with a notion of stability, which in this case will be played by a framing along a line at infinity. We will see how to describe such moduli spaces explicitly via the Beilinson spectral sequence and the so-called ADHM data, which is to say a set of linear maps satisfying commuting relations and a stability condition, which are tantamount to a representation of a certain quiver. 3. Hyper-Kähler and symplectic structure. Here we will review how Hilbert schemes of points on the affine plane inherit a structure of an affine Hyper-Kähler manifold and how this is related to moment maps. On one hand, we will review basic material on Geometric Invariant Theory and on the other hand we will develop a bit the theory of symplectic quotients to show that the moduli spaces obtained by GIT in the previous chapter can actually be described as Hyper-Kahler quotiens. 4. Betti numbers. If time allows, we will get a glimpse of the co-called instanton counting, which is to say, computing the Betti numbers of the moduli spaces seen so far, notably the Hilbert scheme of points on a surface, and study how these numbers obey interesting combinatorial formulas related to partition functions.

## 8. Dissertation M1

The students are required to choose a supervisor and a topic during the month of October and to work on the project under the guidance of the supervisor during the whole academic year. Depending on the actual number of students the project might have to be done in pairs of students.

The work done should be summarised in a dissertation, to be submitted in May.

The general rules for the dissertation are the following:

1. a maximum of 25 pages in TeX,
2. it should not be a direct rephrasing of some chapter of a book,
3. it should contain some personal and detailed take on a specific proof or computation.

Extra material can be included in the appendix without limitation on the number of pages, but won't be evaluated.

The dissertation has to be submitted before the deadline to one of the coordinators and will be evaluated by the supervisor.

The guidelines for evaluation are the following: understanding of the material, quality of the writing, originality in the treatment of the topic, engagement of the student.

The assessment of the supervisor takes into account the work done by the student during the year and the quality of the dissertation.

All the reports will be scanned by an anti-plagiarism software. In the case of evident plagiarism the mark will be zero.

It is not possible to retake the evaluation of the dissertation in the second session.

## 9. Dissertation M2

The students are required to choose a supervisor and a topic during the month of October and to work on the project under the guidance of the supervisor during the whole academic year. For more details please contact the M2 coordinator.

## 10. Exams

The exams of the first semester take place in January, those of the second semester in May, see the calendar above. Typically the exam is a written test which lasts two or three hours.

For the course "Computational methods in mathematical physics" there is also a partial examination during the semester which counts towards 1/3 of the total grade.

**Marks and compensation.** Possible outcomes of an exam are a mark between 0 and 20 or *DEF* (*défaillant*). A course is passed if the mark is greater or equal than 10. *DEF* is attributed to a student that is absent at the final exam, or to a student that is absent at the partial exam and does not have an official justification. Students that are absent at the partial exam but have an official justification will be given a zero mark, unless an extra session is organised, at the discretion of the teacher.

If a student during a semester has one or more exams with a mark less than 10 but the average of the marks of that semester is greater or equal to 10, then those exams are validated "by compensation". Exams are also validated by compensation if the average of the marks of the year is greater or equal to 10. The marks of the exams validated by compensation cannot be improved by retaking the exams later.

If an exam is marked *DEF* then no course in that semester can be validated by compensation.

Exams that are not passed at the first attempt and are not validated by compensation can be retaken in the second session in June.

**Official rules.** The general rules for the exams are detailed in the official document "Référentiel commun des études – Université de Bourgogne", available here:
`www.u-bourgogne.fr/images/stories/odf/ODF-referentiel-etudes-lmd.pdf`

The specific rules for the exams of the Master 1 are detailed in the "Fiche filiére", available here:
`www.u-bourgogne.fr/images/stories/odf/master/ff-mathematical-physics-m1.pdf`

## 11. Repeating the M1

Because the Master receives several outstanding applications each year and the number of available places is limited, students that fail the first year in general will not be allowed to repeat it. Only exceptional circumstances will be considered.

Students that have failed the second session of exams in June and wish to repeat the year should address their application to the coordinators within one week of the publication of the results of the second session.

The students readmitted to the M1 will not benefit from a scholarship.

## 12. F.A.Q.

*I have a mixed background in mathematics/physics/other subject. Do I have any chance to be accepted in the Master?*

As long as you have a background in mathematics or physics, even if you studied other topics, you have a chance to be selected. We cannot say anything in advance however, the only way to know is to apply and let the admission committee examine your dossier.

*How many recommendation letters should I attach to my application? Are they compulsory?*

Recommendation letters are not required, but they can be attached to the application or sent to `math4phys@u-bourgogne.fr` if one wishes so.

*I have been accepted to the master, what should I do?*

You will receive instructions from the administration about the procedure to register and other practical matters.

*When should I confirm my participation to the master?*

We suggest you confirm your participation to (or your withdrawal from) the master as soon as possible, to quickly advance the registration procedures. The formal deadline is in the beginning of September (see the calendar in this Guide). If you are offered a scholarship we require that you confirm your participation within two weeks of the offer.

*At which stage can I apply for a scholarship?*

You will not need to apply specifically for a scholarship, if you are accepted at the Master you will be automatically considered for the scholarship.

*How many students will be funded?*

There number of available scholarships depends on funding which can change every year.

*What are the requirements for the scholarships?*

The selection is based on academic merit, no other criteria are considered.

*What is included in a scholarship?*

The scholarships are around 550 euros per month. Tuition fees are not included.

*What is an estimate of the cost of life for a student in Dijon?*

As an estimate, lodging is around 200 euros/month. Tuition fee is around 240 euros/yr and there is an extra compulsory contribution for 'student life' of 95 euros/yr. See also:
`www.ubfc.fr/en/international/come-to-ubfc/practical-information/`

*When will scholarship recipients be announced?*

Depending on the type of scholarship (we have different sources of financing), recipients will be announced in the months of July-August-September.

*Where can I find other sources of scholarships?*

Unfortunately we are not aware of other sources of funding. We advise international students to enquire at their own institutions for exchanges programmes with France or other funding to study abroad.

*Is it possible to work part-time while following the master?*

Since the schedule of courses is very tight, we do not advise to work alongside the master.

*Do I have to propose a subject for the dissertation or do I have to choose it from a list?*

Usually several topics are proposed by the supervisors by the end of September. We encourage the students to discuss the topics with the supervisors and possibly agree on a new topic before finalising their choice during the month of October.

*I would like to have some material to study during the summer to be more prepared for the beginning of the semester.*

At the moment we don't have a global reading list for the summer for the whole master. Students might contact directly the teachers to ask for preliminary material.

(updated 13/9/2024)